LE MATEMATICHE
Vol. LXII (2007) – Fasc. II, pp. 53–67

HARNACK INEQUALITY FOR HARMONIC FUNCTIONS RELATIVE TO A NONLINEAR P-HOMOGENEOUS RIEMANNIAN DIRICHLET FORM

MARCO BIROLI - PAOLA VERNOLE

We consider a measure valued map $\alpha(u)$ defined on D where D is a subspace of $L^p(X,m)$ with X a locally compact Hausdorff topological space with a distance under which it is a space of homogeneous type.Under assumptions of convexity, Gateaux differentiability and other assumptions on α which generalize the properties of the energy measure of a Dirichlet form, we prove the Holder continuity of the local solution u of the problem $\int\limits_X \mu(u,v)(dx) = 0$ for each v belonging to a suitable space of test functions, where $\mu(u,v) =< \alpha^{'}(u), v >$.

1. Introduction

The aim of the present paper is the proof of a Harnack inequality generalizing the result in [3] and [4] to the nonlinear case.

In [3], [4] Biroli and Mosco proved a Harnack inequality and estimates on the Green function in the case of a Riemannian Dirichlet form (see [3], [4] for the assumptions on the form). For the notion of Dirichlet form we refer to the book of Fukushima-Oshima-Takeda [11]. From Beuerling Deny representation formula [2] a Dirichlet formula is represented as the sum of a strongly local

Entrato in redazione 1 gennaio 2007

AMS 2000 Subject Classification: 31C45, 31C05, 35J60.
Keywords: Nonlinear potential theory, Harnack inequality, Nonlinear elliptic problems

part, of a “killing” part and of a “global part”. The Beuerling Deny representation theorem is the fundamantal tool allowing to prove that the same properties of Dirichlet forms hold again for energy measures in the strongly local (regular case). Using the properties of energy measure of a strongly local Dirichlet form it can be proved that for an energy measure of a strongly local regular Dirichlet form a chain rule and a Leibniz rule hold; those properties are the starting point for an investigation of local regularity of harmonics relative to a strongly local regular Dirichlet form, see [3] [4]. The Beurling-Deny representation theorem is proved using Riesz theorem on representation of measures, which is an essentially linear tool, then it is useless in nonlinear version.

Previous works on a possible extension of the notion of Dirichlet form to the non linear case have been given by Benilan Picard [1], and Cipriani Grillo [8], [9]. The above papers deal with the general global case and are interested in the properties of the corresponding semigroup; then the existence of an energy measure is not ensured and there is no proof of chain or Leibniz rule for the energy measure, when such measure exists. The first paper concerning local forms was [12] where a suitable chain rule for the energy measure connected with the form is assumed and Sobolev-Morrey inequalities are proved as a consequence of a Poincar inequality.

Taking into account that a Riemannian Dirichlet form derives from functional of the type

$$a(u) = \int_X \alpha(u)(dx)$$

where α is a Radon measure homogeneous of degree 2 in u, we consider convex functionals represented by

$$F(u) = \int_X \alpha(u)(dx)$$

where α is a Radon measure and it is homogeneous of degree $p > 1$ in u.

In this case Sobolev type inequality has been proved as consequence of a Poincar inequality by Mosco and Maly [12], so we have one of the main tool in the Moser iteration method.

Our goal is to prove a Harnack inequality and a local Holder continuity result for the nonlinear form connected with the Gateaux derivative of α(u) as operator in the space of Radon measure.

We set $\mu(u,v) = \langle \alpha^{'}(u), v\rangle$ and consider the nonlinear equation

$$\int_X \mu(u,v)(dx) = 0 \tag{1}$$

From the point of view of partial differential equations, the theory we will develop include the p-Laplacian and the subelliptic p-Laplacian. In the first case X is R^n, m is the Lebesgue measure and $\mu(u,v)(dx) = \Sigma|\nabla u|^{p-2} D_{xi}uD_{xi}vdx$ with $D = W^{1,p}(R^n)$; in the second case $X_i = \sum_{j=1}^{n} a_{ij}D_{xj}, \quad i = 1,2...m$ are vector fields defined on R^n with smooth coefficients satisfying a Hormander's condition of length l and $\mu(u,v) = \sum |Xu|^{p-2} X_{iu}X_{iv}dx$. [7]. We observe that also the case of weighted subelliptic p-Laplacian is included in our results.

This is the plan of the paper. In section 2 we recall the definition of linear Dirichlet form and its properties. In section 3 we give the assumptions on the map μ and deduce some useful properties. In section 4 we develop local estimates for solutions and nonnegative subsolutions of a homogeneous problem. In section 5 we prove that a suitable power of every nonnegative supersolution is a weight in the class A_2 of Mukenhoupt [7].

The last section is devoted to the main result; we deduce the Holder continuity of the weak solution of (1) as a consequence of Harnack's inequality [14].Paola Vernole

2. Linear Dirichlet forms

In this section we recall the definition and the properties of linear Dirichlet forms (for details see [11]) Let X be a locally compact, separable, Hausdorff space, and m a nonnegative Radon measure with supp $m = X$. $\;H = L^2(X,m)$.

A Dirichlet form $\mathscr{E}(u,v)$ on H is a nonnegative definite, symmetric bilinear form defined on a dense subspace $D(\mathscr{E})$ in H which is closed, that is $D(\mathscr{E})$ is complete with respect to the norm $\|u\|_D = \left(\mathscr{E}(u,u) + \|u\|_{L^2}^2\right)^{\frac{1}{2}}$,

and further satisfies the (Markovian) property : $u \in D(\mathscr{E}) \Rightarrow v = (0 \vee u) \wedge 1 \in D(\mathscr{E})$ and $\mathscr{E}(v,v) \leq \mathscr{E}(u,u)$.

The Dirichlet form is said strongly local if $\mathscr{E}(u,v) = 0$ for $u,v \in D(\mathscr{E})$ and v constant in a neighborough of the supp u.

The Dirichlet form is said regular if $D(\mathscr{E}) \cap C_0(X)$ is dense in $C_0(X)$ and in $D(\mathscr{E})$.

Any regular and strongly local form $\mathscr{E}$ can be written as

$$\mathscr{E}(u,v) = \int_X \mu(u,v)(dx)$$

where μ is a positive semidefinite, symmetric bilinear form with values in the signed Radon measures on X, uniquely associated to $\mathscr{E}$ (so called energy measure).

3. p-Linear Forms

Let X be a locally compact Hausdorff topological space and m a Radon measure on X with supp $m = X$. Let D be a linear subspace of $L^p(X,m)$ and α a bounded Radon measure valued nonnegative map defined in D.

We make the following assumptions on α.

i) α is positive semidefinite and convex in the space of bounded measure i.e. for each $u \in D \quad \alpha(u)(dx) \geq 0$ and for each $u,v \in D$ a$t \in [0,1]$, $\alpha(tu + (1-t)v)(dx) \leq t\alpha(u)(dx) + (1-t)\alpha(v)(dx)$

ii) α is homogeneous of degree p

iii) α is such that $\|u\|_D = (\int\limits_X \alpha(u)(dx) + \|u\|_{L^p}^p)^{\frac{1}{p}}$ is a norm in D. Moreover we assume that there exists a core i.e a subalgebra C of $D \cap C_0(X)$, which is dense both in C_0 for the uniform norm and in D for the intrinsic norm $\|u\|_D$. By $D_0(A)$ we denote the closure of $D \cap C_0(A)$ in D for the intrinsic norm $\|\|_D$. Furthermore we suppose that the core C has the following separating property:

for every $x, y \in X, x \neq y$, there exists $\varphi \in C$ with $\alpha(\varphi) \leq m$ on X, such that $\varphi(x) \neq \varphi(y)$

iv) Strong locality: if $u - v$ = constant on supp φ, then

$$\int\limits_X \varphi(x)\alpha(u)(dx) = \int\limits_X \varphi(x)\alpha(v)(dx)$$

This property enables us to define the space of functions u that belong locally to the domain of α. If A is *a* relatively compact open subset we denote by $D_{loc}(A)$ the space of functions u defined in A such that exists a function $w \in D$ with $u = w$ in A.

v) Markovianity: if $u \in D$ then $v := (0 \vee u) \wedge 1 \in D$ and $\alpha(v) \leq \alpha(u)$.

We can associate to α a p-capacity set function in the Choquet sense [11] in the following way

$$p - cap(E, \Omega) =$$

$$= inf \left\{ \int\limits_\Omega \alpha(v)(dx), v \in C_0^\infty(\Omega), v \geq 1 \text{ in a neighbourhood of } E \right\}$$

where Ω is an open subset of D and $\bar{E} \subset \Omega$. Every function in D admits a p-quasi-continuous modification $\tilde{u}$; in the following we identify u with its p-quasi-continuous representetive.

vi) α does not charge set of zero capacity

vii) α is Gateaux differentiable i.e. there exists in the weakly* topology the following limit:
$\lim_{t\to 0} \frac{\alpha(u+tv)-\alpha(u)}{t} = \langle \alpha^{'}(u), v\rangle$. We define $\mu : D \times D \longrightarrow$ space of bounded Radon measure $\mu(u,v) = \langle \alpha'(u), v\rangle$.

viii) chain rules $\beta \in C^1(\Re)$ with $\beta(0) = 0$ then $\beta(u), \beta(v) \in D_0$ and

$$\mu(\beta(u), v) = \left|\beta^{'}(u)\right|^{p-2} \beta^{'}(u)\mu(u,v)$$

$$\mu(u, \beta(v)) = \beta^{'}(v)\mu(u,v)$$

From the locality and the chain rules it follows the truncation property. Namely for every u and $v \in D$

$$\mu(u^+, v) = 1_{\{\tilde{u}>0\}}\mu(u,v)$$

$$\mu(u, v^+) = 1_{\{\tilde{v}>0\}}\mu(u,v)$$

where $\tilde{u}$ and $\tilde{v}$ are the quasi continuous versions of u and v respectively.

From the definition of μ and the assumptions on α we get some important properties on μ we summarize in the following

Proposition 3.1. *For any u, v belonging to $D \cap L^\infty(X,m)$ we have:*

i) $\mu(u,v)$ is homogeneous of degree $p-1$ in u and linear in v

ii) $\forall a \in R^+$

$$|\mu(u,v)| \leq \alpha(u+v) \leq 2^{p-1}a^{-p}\alpha(u) + 2^{p-1}a^{p(p-1)}\alpha(v) \tag{2}$$

iii) $\alpha(u,u) = p\alpha(u)$

iv) Leibniz rule on the second argument

$$\forall w \in D \cap L^{\infty}(X,m) \mu(u,vw) = v\mu(u,w) + w\mu(u,v) \tag{3}$$

v) For any $f \in L^{p'}(X,\alpha(u))$ and $g \in L^{p}(X,\alpha(v))$ with $\frac{1}{p}+\frac{1}{p'}=1$, fg is integrable with respect to the absolute variation of $\mu(u,v)$ and $\forall a \in R^{+}$

$$\int_X |fg|\,|\mu(u,v)|\,(dx) \leq$$

$$2^{p-1}a^{-p}\int_X |f|^{p'}\alpha(u)(dx) + 2^{p-1}a^{p(p-1)}\int_X |g|^{p}\alpha(v)(dx) \tag{4}$$

For the proof see [5]

Thanks to the separating property of the core C we can consider on X a metric $d : X \times X \to R^{+}$ induced by μ in the following way:

$$d(x,y) = \sup\{\varphi(x) - \varphi(y) : \varphi \in C, \alpha(\varphi) \leq m \quad onX\}$$

and assume that (X,d) is a space of homogeneous type i.e.

a) the metric topology induced by d is equivalent to the original topology of X.

b) denoted by $B(x,r)$ the quasiball of center x and radius r there exist positive constants c_0 and r_0 such that

$$0 < m(B(x,2r)) \leq c_0 m(B(x,r)) \quad \forall x \in X \ \ and \ \ 0 < r < r_0 \tag{5}$$

The assumption b) implies that there are constants $c_1 > 0$ and $\nu > 0$ such that

$$m(B(x,r)) \leq c_1 m(B(x,s)) \left(\frac{r}{s}\right)^{\nu} \quad \forall x \in X \ \ and \ \ 0 < s < r < r_0 \tag{6}$$

We recall a covering property of homogeneous spaces ([10]).

Lemma 3.2. *Let B(x,R), $0 < R < r_0$,be a ball of X: For arbitrary $r \in (0,R)$, there exists a sequence of balls $B(y_i,r), y_i \in B(x,R), i = 1,2,...,$ Q with $Q = c\left(\frac{R}{r}\right)^\nu$ such that $B(x,R) \subset \cup B(y_i,r)$*

We need a relationship between the metric and measure in X and the measure valued map α. We assume that, given a relatively compact subset A of X, there exist constants $c_2 > 0$ and $k \geq 1$ such that for every $x \in A$ and every $r > 0$ with $B(x,r) \subset A$ the following Poincar inequality of order p holds

$$\int_{B(x,r)} |u - \bar{u}_r|^p \, dm \leq c_2 r^p \int_{B(x,kr)} \mu(u,u)(dx) \tag{7}$$

for every $u \in D_{loc}(A)$, where $\bar{u}_r = [m(B(x,r)]^{-1} \int_{B(x,r)} udm.$

Let us assume $p < \nu$. If A is connected, the assumption of homogeneity for the space X and the Poincar inequality imply the following inequality

$$\left(\frac{1}{m(B(x,r))} \int_{B(x,r)} |u|^{p^*} \, dm\right)^{\frac{1}{p^*}} \leq$$

$$\leq c\left(\frac{r^p}{m(B(x,r))} \int_{B(x,kr)} \mu(u,u)(dx) + \frac{1}{m(B(x,r))} \int_{B(x,r)} |u|^p \, dm\right)^{\frac{1}{p}} \tag{8}$$

with $p^* = \frac{p\nu}{\nu - p}$ and c depending only on c_0 and c_2. So we have the analogous of classical Sobolev imbedding theorem [12].

Thanks to the definition of d we can construct ”cut off” functions like in the classical frame.It is possible to prove the following

Lemma 3.3. *Given any two concentric balls $B(x,s)$ and $B(x,t)$ with $0 < s < t < r_0$ there exists a function $\varphi \in C_0^\infty(R^n)$ such that $0 \leq \varphi \leq 1, \varphi(y) = 1 \;\; \forall y \in B(x,s)$, supp $\varphi \subset B(x,t)$ and $\alpha(\varphi) \leq c/(t-s)^p m$.*

Remark 3.4. If $u \in D_0(B(x,r))$ the inequality (8) holds without the presence of the term

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |u|^p \, dm$$

4. Local L^∞-estimates

Definition 4.1 Let Ω be an open subset of X. $u \in D_{loc}(\Omega)$ is a local solution of (1) if

$$\int_\Omega \mu(u,v)(dx) = 0 \quad for \;\; every \;\; v \in D_0(\Omega)$$

$u \in D_{loc}(\Omega)$ is a subsolution (supersolution) if

$$\int_\Omega \mu(u,v)(dx) \le 0 (\ge 0) \;\; for \;\; every \;\; v \in D_0(\Omega) \;\; v \ge 0m \;\; a.e. \;\; in \Omega.$$

It is possible to prove a maximum principle for the solution or the subsolution of our problem, for the proof see [6]

Proposition 4.1. *Let A be an open subset of some ball $B \subset \Omega$. Let $u \in D_{loc}(X) \cap C^\circ(\bar{A})$ be such that*

$$\int_A \mu(u,v)(dx) \le 0, \quad \forall v \in D_0(A), \;\; v \ge 0 \;\; m \;\; a.e. \;\; in \;\; A.$$

Then

$$u(x) \le \max_{\partial A} u \;\; \forall x \in A$$

Also in our case a well known Caccioppoli type inequality holds

Proposition 4.2. *Let A be an open bounded subset of Ω and $B_s \subset B_t \subset \bar{B}_{2t} \subset A$ be concentric balls, $0 < s < t$. Let $u \in D_{loc}(A)$ be a local solution (positive subsolution) in A of the problem $a(u,v) = 0, \forall v \in D_0(A)$.Then*

$$\int_{B_s} \mu(u,u)(dx) \le \frac{c}{(t-s)^p} \int_{B_t - B_s} |u|^p \, dm$$

The previous propositions allow us to prove a lemma and a theorem on the subsolution which will be useful to prove Harnack inequality.

Lemma 4.3. *Let u be a local positive subsolution in an open bounded set A and $\beta > 0$. Let $\eta \in C_0^\infty(A), \eta \ge 0$. Then*

$$\int_A \eta^p u^{\beta-1} \mu(u,u)(dx) \le C(\beta,p) \int_A u^{\beta+p-1} \alpha(\eta)(dx)$$

Theorem 4.4. *Let u be a positive subsolution on $B(x,8r) \subset A$ $q > p-1$. Then there exist constants C_q and L such that for each s and t with $\frac{1}{2} \leq s < t \leq 1$*

$$\sup_{B(x,sr)} u \leq \frac{C_q}{(t-s)^L} \left(\frac{1}{m(B(x,tr))} \int_{B(x,tr)} u^q dm \right)^{\frac{1}{q}} \tag{9}$$

We recall now a lemma of real analysis, for the proof see [4] Lemma 5.2.

Lemma 4.5. *Let $u \in L^\infty(B(x,r),m)$ and assume there exist positive constants C, d and L such that for every s,t with $\frac{1}{2} \leq s < t \leq 1$ we have*

$$\sup_{B(x,sr)} u \leq \frac{C}{(t-s)^L} \left(\frac{1}{m(B(x,tr))} \int_{B(x,tr)} u^{2d} dm \right)^{\frac{1}{2d}} .$$

Then for every $a > 0$ there exists a constant C_a, which depends on a, C, d and on the constant c_0,such that

$$\sup_{B(x,\frac{r}{2})} u \leq C_a \left(\frac{1}{m(B(x,r))} \int_{B(x,r)} u^a dm \right)^{\frac{1}{a}}$$

Theorem 4.6. *Let u be a positive subsolution on $B(x,8r) \subset A$. Then, for every $q > 0$ there exists a constant C_q such that*

$$\sup_{B(x,r)} u \leq C_q \left(\frac{1}{m(B(x,tr))} \int_{B(x,2r)} u^q dm \right)^{\frac{1}{q}} . \tag{10}$$

Proof. Thanks to theorem 4.5 we have inequality 9 for every $q > p-1$. Taking into account lemma 4.6 we get that inequality 10 holds for every $q > 0$. $\square$

5. Functions of Bounded mean oscillation and their properties

Definition 5.1 A function u on X is said of bounded mean oscillation (BMO) on an open subset E of X, if $u \in L^1(E,m)$ and the following seminorm is finite

$$\|u\|_{BMO} = sup \left\{ [m(B)]^{-1} \int_B |u - \bar{u}| \, dm \right\}$$

where $\bar{u}$ denotes the average $\bar{u} = [m(B)]^{-1} \int_B udm$ and the supremum is taken over the family of all balls contained in E. Now we recall a corollary on functions of bounded mean oscillation that will be useful in the sequel. For the proof see [4] corollary 5.6.

Corollary 5.1. *Let $B(x_0, 12r) \subset X$, $0 < 12r < r_0$, and let u be a fuction of bounded mean oscillation. Then for every $B = B(x,r)$ with $x \in B(x_0,r)$ and $M > \|u\|_{MBO,B(x_0,12r)}$ there exists $A \geq 1$ and $\alpha > 0$ depending only on the constant c_0, such that*

$$[m(B)]^{-2} \int_B exp(\frac{\alpha}{2M}u)dm \int_B exp(-\frac{\alpha}{2m}u)dm \leq A$$

We consider an open subset A of Ω and a function v which satisfies:

$$v \in D_{loc}(A) \cap L^\infty(A,m), \;\; v \geq 0 \;\; m \;\; a.e \;\; inA \;\; a(v,w) \geq 0 \qquad (11)$$

$$\forall w \in D_0(A) \;\; w \geq 0 \;\; m \;\; a.e \;\; in \;\; A$$

Proposition 5.2. *Let v satisfies* 9 *and let $B(x.4r) \subset A$.Then*

$$\int_{B(x,r)} \mu(\log(v+\varepsilon), \log(v+\varepsilon))(dx) \leq c\frac{m(B(x,r))}{r^p}$$

for every $\varepsilon > 0$.

Now we are able to prove that $log(v+\varepsilon)$ is a function of bounded mean oscillation on every ball $B(x,r)$ such that $B(x,4r) \subset A$

Lemma 5.3. *Let $B(x,4r) \subset A$ and v satisfies* 10*. Then, for every $\varepsilon > 0$*

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |\log(v+\varepsilon) - (\log(v+\varepsilon))_r|^p \, dm \leq C$$

where $(log(v+\varepsilon))_r$ is the average of $\log(v+\varepsilon)$ on the ball $B(x,r)$ and C is a constant depending only on c_0, c_2, p, k. From Lemma 4.1 it follows

Theorem 5.4. *Let $B(x,r)$ be an arbitrary ball in $B(x_0,12r)$. Then $log(v+\varepsilon)$ is a BMO fuction in $B(x_0,12r)$ and*

$$\|log(v+\varepsilon)\|_{BMO(B(x_0 12r))} \leq c$$

Now we are able to prove that a suitable power of any non negative supersolution is a weight in the class A_2 of Mukenhoupt.

Proposition 5.5. *Let $v \in D_{loc}(A) \cap L^\infty(A,m)$ be a non negative supersolution, let $B(x_0,16R) \subset A$ and let $x \in B(x_0,R)$ with $0 < r \leq R$ and $v > 0$ m a.e. in $B(x,r)$. Then there exist $\gamma \in (0,1)$ depending on c_0, c_2 and $A \geq 1$ depending on c_0 such that*

$$\int\limits_{B(x,r)} v^\gamma dm \int\limits_{B(x,r)} v^{-\gamma} dm \leq A \tag{12}$$

For the proofs see [6]

6. Harnack inequality

In this section we will prove the main theorem of our paper Harnack inequality for the nonnegative local solution of equation (1)

First of all we will prove a lemma about the behaviour of the power of nonnegative solution of (1)

Lemma 6.1. *Let u be a nonnegative solution of (1) in $B(x,4r)$. Then the function u^q belongs to $D_{loc}B(x,4r) \cap L^\infty(B(x,4r),m)$ and is a positive subsolution in $B(x,4r)$ if $q<0$ or $q>1$, while is a supersolution if $q \in (0,1)$.*

Proof. We set $w = (u+\varepsilon)^q$ where $\varepsilon > 0$. By the chain rules if $z \in D_0(B(x,4r) \cap L^\infty(B(x,4r),m)$ and $z \geq 0$ we have:

$$a(w,z) = \int\limits_{B(x,4r)} \mu(w,z)(dx) = q|q|^{p-2} \int\limits_{B(x,4r)} (u+\varepsilon)^{(q-1)(p-1)} \mu((u+\varepsilon),z)(dx)$$

and taking into account the Leibniz rule

$$= q|q|^{p-2} \left[\int\limits_{B(x,4r)} \mu(u+\varepsilon,(u+\varepsilon)^{(q-1)(p-1)} z)(dx) \right] +$$

$$-q\,|q|^{p-2}\left[\int\limits_{B(x,4r)} z(q-1)(p-1)(u+\varepsilon)^{pq-q-p}\mu(u+\varepsilon,u+\varepsilon)(dx)\right]$$

Since u is a solution also $u+\varepsilon$ is a solution so the first integral is zero and $a(w,z)\leq 0$ if $-q(q-1)\leq 0$ i.e. $q\leq 0$ or $q\geq 1$; while $a(w,z)\geq 0$ if $q\in(0,1)$. Letting ε go to zero, by the monotone convergence theorem, we obtain the thesis. □

Now we are able to prove the Harnack inequality for the local solution of (1).

Theorem 6.2. *Let u be a positive local solution of* (1) *in A. Then for every ball $B(x,r)\subseteq B(x,8r)\subseteq A$ we have*

$$\sup_{B(x,r)} u \leq C \inf_{B(x,r)} u$$

where C is a constant depending on c_0, c_2, p.

Proof. Let u be a local positive solution of (1), by the previous lemma we know that $w=u^{-1}$ is a positive subsolution. We can thus apply theorem 4.7 and we find

$$\sup_{B(x,r)} w \leq C_q\left(\frac{1}{m(B(x,2r))}\int\limits_{B(x,2r)} w^q dm\right)^{\frac{1}{q}}$$

$$\inf_{B(x,r)} u \geq C_q^q\left(\frac{1}{m(B(x,2r))}\int\limits_{B(x,2r)} u^{-q} dm\right)^{-\frac{1}{q}}$$

Now we choose $q=\gamma$ where γ is the constant of proposition 5.6 and suppose that $B(x,64r)\subset A$. Then taking into account 12

$$\inf_{B(x,r)} u \geq C_\gamma\left(\frac{1}{m(B(x,2r))}\int\limits_{B(x,2r)} u^{-\gamma} dm\right)^{-\frac{1}{\gamma}} \geq$$

$$\geq CA^{-\frac{1}{\gamma}} \left(\frac{1}{m(B(x,2r))} \int_{B(x,2r)} u^{\gamma} dm \right)^{\frac{1}{\gamma}} \geq C \sup_{B(x,r)} u \tag{13}$$

□

Theorem 6.3. *Let $u \in D_{loc}(A)$ be a local solution of* 1 *where A is an open subset of Ω. Then u is Holder continuous*

Proof. We can apply theorem 13 to the function $u - \inf_{B(x,4r)} u$ for any ball $B(x,r)$ such that $B(x,4r) \subset A$ and we get:

$$\sup_{B(x,r)} \left[u - \inf_{B(x,4r)} u \right] \leq C \inf_{B(x,r)} \left[u - \inf_{B(x,4r)} u \right] \tag{14}$$

Let us set $M(r) =: \sup_{B(x,r)} u \quad m(r) =: \inf_{B(x,r)} u \quad$ 14 becomes

$$M(r) - m(4r) \leq C\,(m(r) - m(4r)) \tag{15}$$

Set $\Lambda = \frac{C-1}{C}$ it is easy to prove that

$$osc(u, B(x,r)) \leq \Lambda osc(u, B(x,4r)) \tag{16}$$

Iterating 15 we complete the proof.[13] □

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MARCO BIROLI
Dipartimento di Matematica "F. Brioschi"
Politecnico di Milano,
Piazza Leonardo da Vinci 32, 20133 Milano Italy
e-mail: `marbir@mate.polimi.it`

PAOLA VERNOLE
Dipartimento di Matematica
University of Rome "la Sapienza",
Piazzale Aldo Moro 5, 00182 Roma Italy
e-mail: `vernole@mat.uniroma1.it`